# The Cuntz semigroup of the crossed product by a tracially strictly approximately inner action

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August 11, 2020

Special Week on Operator Algebras 2020

Research Center for Operator Algebras of East China Normal University

August 11, 2020

This material is based on the following work:

1. [A20] M. Ali Asadi-Vasfi, The radius of comparison of the crossed product by a tracially strictly approximately inner action, preprint (arXiv: 2007.09783v1 [math.OA]).

Warning: Some definitions are not stated carefully.

# Introduction

Let $A$ be a C*-algebra and let $p, q \in A$ be projections. We say $p$ is Murray-von Neumann subequivalent to $q$, denoted $p \precsim q$, if there exists $v \in A$ such that $p = vv^*$ and $v^*v \leq q$. We say that $p$ and $q$ are Murray-von Neumann equivalent, denoted $p \approx q$, if there exists $v \in A$ such that $p = vv^*$ and $v^*v = q$.

The Murray-von Neumann semigroup $V(A)$ of $A$ is defined as the set of equivalence classes of projections in matrices over $A$.

The problem with V($A$):

If $A$ has few or no projections, then $V(A)$ may say nothing about the structure of the C*-algebra.

Question:

What is an appropriate replacement for V($A$)?

The appropriate replacement for V($A$) was first introduced by Cuntz in 1978 and, later, by Coward-Elliott-Ivanescu in 2008.

The idea of Cuntz was to consider positive elements instead of projections.

## Definition

Let $A$ be a C*-algebra. Let $m, n \in \mathbb{Z}_{>0}$, let $a \in M_n(A)_+$, and let $b \in M_m(A)_+$.

1. We say that $a$ is Cuntz subequivalent to $b$ in $A$, written $a \precsim_A b$, if there exists a sequence $(x_k)_{k=1}^{\infty}$ in $M_{n,m}(A)$ such that

$$\lim_{k \to \infty} x_k b x_k^* = a.$$

2. We say that $a$ is Cuntz equivalent to $b$ in $A$, written $a \sim_A b$, if $a \precsim_A b$ and $b \precsim_A a$.

## Example

Let $n \in \mathbb{Z}_{>0}$, let $A = M_n\big(C([0,1])\big)$, and let $f, g \in A_+$. Then

$$f \precsim_A g \iff \operatorname{rank}\left(f(t)\right) \leq \operatorname{rank}\left(g(t)\right) \quad \text{for all } t \in [0,1].$$

## Example

Let $X$ be a compact metric space and let $f, g \in C(X)_+$. Then

$$f \precsim_{C(X)} g \Longleftrightarrow \{x \in X \colon f(x) \neq 0\} \subseteq \{x \in X \colon g(x) \neq 0\}.$$

## Definition

We denote by $M_\infty(A)$ the algebraic limit of the direct system $(M_n(A), \varphi_n)$ where $\varphi_n \colon M_n(A) \to M_{n+1}(A)$ is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Also define

$$\mathrm{W}(A) = M_\infty(A)_+ / \sim_A .$$

Define the operation $\langle a \rangle_A + \langle b \rangle_A = \langle a \oplus b \rangle_A$ and the partial order $\langle a \rangle_A \leq \langle b \rangle_A$ if $a \precsim_A b$. $W(A)$ is an ordered abelian monoid. The *Cuntz semigroup* of $A$ is

$$\mathrm{Cu}(A) = \mathrm{W}(\mathcal{K} \otimes A).$$

**Example**

Let $A = C([0,1])$. Then

1. $\mathrm{V}(A) = \mathbb{Z}_{>0}$
2. $\mathrm{Cu}(A) = \{f\colon [0,1] \to \mathbb{Z}_{>0} \cup \{\infty\} | \ f$ is lower semicontinuous$\}$.

We let $QT(A)$ be the set of all normalized quasitraces on $A$, (without renormalization: the quasitrace of the identity of $M_n(A)$ is $n$, not 1).

## Definition

Let $A$ be a stably finite unital C*-algebra.

1. For every $\tau \in \mathrm{QT}(A)$ and every $a \in \bigcup_{k=1}^{\infty} M_k(A)_+$, define

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}).$$

2. For $r \in [0, \infty)$, $A$ has $r$-comparison if whenever $a, b \in \bigcup_{k=1}^{\infty} M_k(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim_A b$.
3. The radius of comparison of $A$, denoted $\mathrm{rc}(A)$, is

$$\mathrm{rc}(A) = \inf\left(\left\{r \in [0, \infty) \colon A \text{ has } r\text{-comparison}\right\}\right).$$

(We take $\mathrm{rc}(A) = \infty$ if there is no $r$ such that $A$ has $r$-comparison.)

If $A = C(X)$, the condition $d_\tau(a) + r < d_\tau(b)$ for all $\tau$ becomes

$$\operatorname{rank}(a(x)) + r < \operatorname{rank}(b(x))$$

for all $x$ in $X$.

**Theorem (Blackadar-Robert-Tikuisis-Toms-Winter 2012)**

Let $X$ be a compact metric space. Then

$$\operatorname{rc}\big(C(X)\big) \leq \frac{1}{2}\dim(X).$$

**Proposition (Toms 2006)**

Let $A$ and $B$ be stably finite unital C*-algebras. Then:

1. $\operatorname{rc}(A \oplus B) = \max\big(\operatorname{rc}(A), \operatorname{rc}(B)\big)$.
2. If $k \in \mathbb{Z}_{>0}$, then $\operatorname{rc}\big(M_k(A)\big) = \frac{1}{k} \cdot \operatorname{rc}(A)$.

**Theorem (rc of the corner)**

Let $A$ be a stably finite unital C*-algebra and let $p$ be a full projection in $A$. Define

$$\lambda = \inf\big(\{\tau(p)\colon \tau \in \mathrm{QT}(A)\}\big) \qquad \text{and} \qquad \eta = \sup\big(\{\tau(p)\colon \tau \in \mathrm{QT}(A)\}\big).$$

Then $0 < \lambda \leq \eta \leq 1$ and

$$\frac{1}{\eta} \cdot \mathrm{rc}(A) \leq \mathrm{rc}(pAp) \leq \frac{1}{\lambda} \cdot \mathrm{rc}(A).$$

**Example (Motivating example)**

Let $A$ be a stably finite unital C*-algebra. Let $\alpha\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(A)$ be an inner action. Then

$$C^*(G, A, \alpha) \cong C^*(\mathbb{Z}/2\mathbb{Z}) \otimes A \cong (\mathbb{C} \oplus \mathbb{C}) \otimes A \cong A \oplus A.$$

Therefore

$$\mathrm{Cu}\left(C^*(G, A, \alpha)\right) \cong \mathrm{Cu}(A) \oplus \mathrm{Cu}(A)$$

and

$$\mathrm{rc}\big(C^*(G, A, \alpha)\big) = \mathrm{rc}(A \oplus A) = \mathrm{rc}(A).$$

## Definition

Let $A$ be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group $G$ on $A$. The action $\alpha$ has the Rokhlin property if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $p_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|p_g a - a p_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g\in G} p_g = 1$.

One can generalize the above definition to weak tracial Rokhlin property. The idea is to replace Rokhlin projections by positive contractions and replace condition (3) by $1 - \sum_{g\in G} p_g$ being small in the Cuntz semigroup.

## Definition

Let $A$ be a unital C*-algebra, let $G$ be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of $G$ on $A$. We say that $\alpha$ is approximately representable if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are $z_g \in U(A)$ for $g \in G$ such that:

1. $\|\alpha_g(a) - z_g a z_g^*\| < \varepsilon$ for all $g \in G$ and $a \in F$.
2. $\|z_g z_h - z_{gh}\| < \varepsilon$ for all $g, h \in G$.
3. $\|\alpha_g(z_h) - z_{ghg^{-1}}\| < \varepsilon$ for all $g, h \in G$.

Izumi proved that if $G$ is a finite abelian group and $\alpha\colon G \to \mathrm{Aut}(A)$ is an approximate representable action, then $\widehat{\alpha}$ is an action of $\widehat{G}$ on $C^*(G, A, \alpha)$ with the Rokhlin property.

Now, we omit Condition (3) in the previous definition and define the following.

**Definition**

Let $A$ be a unital C*-algebra, let $G$ be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of $G$ on $A$. We say that $\alpha$ is strictly approximately inner if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are $z_g \in U(A)$ for $g \in G$ such that:

1. $\|\alpha_g(a) - z_g a z_g^*\| < \varepsilon$ for all $g \in G$ and $a \in F$.
2. $\|z_g z_h - z_{gh}\| < \varepsilon$ for all $g, h \in G$.

Obviously, approximate representability implies strict approximate innerness. There is no reason to expect that the converse in true. We also have the tracial analog of the above notion under the name "tracially strictly approximately inner".

## Notation

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group $G$ on a unital C*-algebra $A$. For each $g \in G$, we define the map $E_g\colon C^*(G, A, \alpha) \to A$ by

$$E_g(a) = a_g,$$

where $a = \sum_{g \in G} a_g u_g$. We also denote by $A^\alpha$ the fixed point algebra, given by

$$A^\alpha = \left\{ a \in A\colon \alpha_g(a) = a \text{ for all } g \in G \right\}.$$

# Warning:

From now, we focus on basic methods and basic ideas rather than technical aspects in the proofs of theorems, propositions, etc.
In a number of cases, we restrict our attention to the actions which are strictly approximately inner. In some cases, the proofs of the tracial versions are more difficult and technical.

## Theorem ([A20])

Let $A$ be an infinite-dimensional simple unital C*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group $G$ on $A$ which is tracially strictly approximately inner. Then for every finite set $F \subset C^*(G, A, \alpha)$, every $\varepsilon > 0$, and every $a \in A_+$ with $\|a\| = 1$, there are a projection $e \in A^\alpha$ and a unital surjective linear map $\psi\colon eC^*(G, A, \alpha)e \to eAe$ such that:

1. $\|\psi\| \leq \mathrm{card}(G)$.
2. $\psi(ebe) = ebe$ for all $b \in A$.
3. $\|\psi(exyz^*e) - \psi(exe)\psi(eye)\psi(eze)^*\| < \varepsilon$ for all $x, y, z \in F$.
4. $1 - e \precsim_A a$.
5. $\|eae\| > 1 - \varepsilon$.

## Sketch of the proof.

For simplicity, we assume that $\alpha$ is strictly approximately inner. - - →

## Sketch of the proof (continued).

Let $\varepsilon > 0$ and let $F \subset C^*(G, A, \alpha)$ be a finite set. Set

$$F_0 = \Big\{ E_g(v), E_g(v)^* \colon g \in G \text{ and } v \in F \Big\},$$

$$M = \max\Big(1,\ \max_{a \in F_0} \|a\|\Big), \qquad \text{and} \qquad \varepsilon' = \frac{\varepsilon}{2M^2 \operatorname{card}(G)^3}.$$

Since $\alpha$ is strictly approximately inner, we get a homomorphism $g \mapsto z_g$ from $G$ to $U(A)$ such that, for all $g \in G$ and $a \in F_0$,

$$\|\alpha_g(a) - z_g a z_g^*\| < \varepsilon'.$$

Now define $\psi \colon C^*(G, A, \alpha) \to A$ by

$$\sum_{g \in G} a_g u_g \mapsto \sum_{g \in G} a_g z_g.$$

Clearly $\psi$ is unital, linear, and surjective. We can show that $\psi$ is an approximate homomorphism on $F$. ■

### Theorem ([A20])

Let $A$ be an infinite dimensional simple unital C*-algebra, let $a, b \in A_+$, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group $G$ on $A$ which is tracially strictly approximately inner. Assume 0 is a limit point of $\mathrm{sp}(b)$. Then

$$a \precsim_{C^*(G,A,\alpha)} b \quad \text{if and only if} \quad a \precsim_A b.$$

### Sketch of the proof.

For simplicity, we assume that $\alpha$ is strictly approximately inner. The proof of the full version requires more work to deal with the tracial error. $\dashrightarrow$

## Sketch of the proof (continued).

Let $\varepsilon > 0$. Let $a, b \in A_+$. We use $a \precsim_{C^*(G,A,\alpha)} b$ to find $v \in C^*(G, A, \alpha)$ such that

$$\|vbv^* - a\| < \frac{\varepsilon}{2\mathrm{card}(G)}.$$

Set $F = \{b, v, v^*\}$. Now, we apply the previous theorem with $\frac{\varepsilon}{2}$ and $F$ as given to get a unital surjective linear map $\psi\colon C^*(G, A, \alpha) \to A$ such that:

1. $\|\psi\| \leq \mathrm{card}(G)$.
2. $\psi(c) = c$ for all $c \in A$.
3. $\|\psi(xyz^*) - \psi(x)\psi(y)\psi(z)^*\| < \frac{\varepsilon}{2}$ for all $x, y, z \in F$.

We can show that

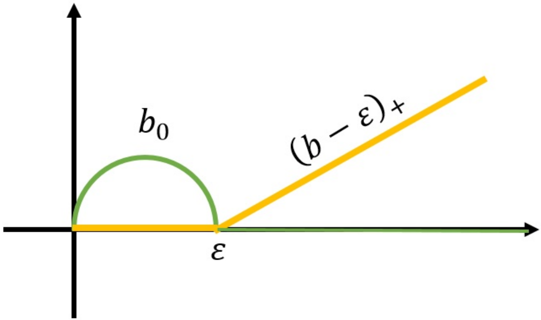
$$\|a - \psi(v)b\psi(v)^*\| < \varepsilon.$$

Therefore

$$(a - \varepsilon)_+ \precsim_A \psi(v)b\psi(v)^* \precsim_A b.$$

■

If $\alpha$ is strictly approximately inner, the assumption that 0 is a limit point of sp($b$) is not necessary. The problem arises only with the tracial version of strict approximate innerness. In that case, if $0 \notin$ sp($b$), then $vbv^*$ does not cover all of $a$ in trace, and something besides $(b - \varepsilon)_+$ is needed to take care of the part of $a$ that is missed. That is what $b_0$ does.



## Notation

Let $A$ be a C*-algebra. We define

$$A_{++} = \big\{ a \in A_+ \colon \text{there is no projection } q \in M_\infty(A) \text{ such that } \langle a \rangle_A = \langle q \rangle_A \big\}.$$

and

$$\mathrm{Cu}_+(A) = \big\{ \langle a \rangle_A \colon a \in (\mathcal{K} \otimes A)_{++} \big\}.$$

The elements of $A_{++}$ are called *purely positive*.

## Remark

If $A$ is a stably finite simple unital C*-algebra, then

$$(\mathcal{K} \otimes A)_{++} = \big\{ a \in (\mathcal{K} \otimes A)_+ \colon 0 \text{ is a limit point of } \mathrm{sp}(a) \big\},$$

and $\mathrm{Cu}_+(A) \cup \{0\}$ is a unital subsemigroup of $\mathrm{Cu}(A)$.

## Theorem ([A20])

Let $A$ be a unital C*-algebra and let $G$ be a finite group. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Let $\iota\colon A \to C^*(G, A, \alpha)$ be the inclusion map. Then:

1. If $\alpha$ is strictly approximately inner, then the map
$$\mathrm{Cu}(\iota)\colon\ \mathrm{Cu}(A) \to \mathrm{Cu}(C^*(G, A, \alpha))$$
is an ordered semigroup isomorphism onto its range.
2. If $\alpha$ is tracially strictly approximately inner and $A$ is stably finite simple C*-algebra which is not of type I, then the map
$$\mathrm{Cu}(\iota)\colon\ \mathrm{Cu}(A) \to \mathrm{Cu}(C^*(G, A, \alpha))$$
induces an isomorphism of ordered semigroups from $\mathrm{Cu}_+(A) \cup \{0\}$ to its image in $\mathrm{Cu}(C^*(G, A, \alpha))$.

# Application:

**Definition**

An ordered semigroup $S$ is said to be almost unperforated if for every $x, y \in S$ and $n \in \mathbb{Z}_{>0}$ such that $(n+1)x \leq ny$, then $x \leq y$.

**Corollary ([A20])**

Let $A$ be a stably finite simple unital C*-algebra which is not of type I and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group $G$ on $A$ which is tracially strictly approximately inner. If $\mathrm{Cu}_+\left(C^*(G, A, \alpha)\right) \cup \{0\}$ is almost unperforated, then so is $\mathrm{Cu}(A)$.

## Theorem ([A20])

Let $A$ be an infinite-dimensional simple unital stably finite C*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a tracially strictly approximately inner action of a finite group $G$ on $A$. Then:

1. $\mathrm{rc}(A) \leq \mathrm{rc}\big(C^*(G, A, \alpha)\big)$.
2. If $C^*(G, A, \alpha)$ is simple, then

$$\mathrm{rc}(A) \leq \mathrm{rc}\big(C^*(G, A, \alpha)\big) \leq \mathrm{rc}(A^\alpha).$$

## Sketch of the proof.

For simplicity, we assume that $\alpha$ is strictly approximately inner. The proof of the full version requires more work to deal with the tracial error. --→

## Sketch of the proof (continued).

Assume $\mathrm{rc}\big(C^*(G,A,\alpha)\big) < \infty$. Let $r \in [0,\infty)$. Suppose that $C^*(G,A,\alpha)$ has $r$-comparison. Let $a, b \in M_\infty(A)_+$ satisfy

$$d_\rho(a) + r < d_\rho(b)$$

for all $\rho \in \mathrm{QT}(A)$. Since every quasitrace on $C^*(G,A,\alpha)$ restricts to a quasitrace on $A$, it follows that

$$d_\rho(a) + r < d_\rho(b)$$

for all $\rho \in \mathrm{QT}(C^*(G,A,\alpha))$. Now, since $C^*(G,A,\alpha)$ has $r$-comparison, we get $a \precsim_{C^*(G,A,\alpha)} b$. Then $a \precsim_A b$. Therefore $\mathrm{rc}(A) \leq r$. Taking the infimum over $r \in [0,\infty)$ such that $C^*(G,A,\alpha)$ has $r$-comparison, we get

$$\mathrm{rc}(A) \leq \mathrm{rc}\big(C^*(G,A,\alpha)\big).$$

■

# Applications:

In the following proposition, we show that actions of finite groups on many infinite-dimensional simple nonclassifiable C*-algebras cannot simultaneously have the weak tracial Rokhlin property and be tracially strictly approximately inner.

## Proposition ([A20])

Let $A$ be an infinite-dimensional simple unital stably finite C*-algebra with $0 < \mathrm{rc}(A) < \infty$. Then there is no action of any nontrivial finite group on $A$ which both has the weak tracial Rokhlin property and is tracially strictly approximately inner.

**Proof.**

Assume that $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a nontrivial finite group $G$ on $A$ which both has the weak tracial Rokhlin property and is tracially strictly approximately inner. Then, by the result of [AGP19] and the above theorem,

$$\mathrm{rc}(A) \leq \mathrm{rc}(C^*(G, A, \alpha)) \leq \frac{1}{\mathrm{card}(G)} \cdot \mathrm{rc}(A).$$

This is a contradiction. ∎

**Proposition (Toms 2006)**

Let $A$ be an unital and stably finite C*-algebra for which every $\tau \in \mathrm{QT}(A)$ is faithful. If $\mathrm{rc}(A) = 0$, then $\mathrm{W}(A)$ is almost unperforated.

**Corollary ([A20])**

Let $A$ be an infinite-dimensional simple unital stably finite C*-algebra with $\mathrm{rc}(A) < \infty$ and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a (nontrivial) finite group $G$ on $A$. If $\alpha$ is tracially strictly approximately inner and has the weak tracial Rokhlin property. Then $\mathrm{Cu}(A)$ is almost unperforated.

We expect that much more might be true in the above corollary. Namely, almost unperforation can be replaced by tracial $\mathcal{Z}$-stability of $A$.

## Theorem ([A20])

For every finite group $G$ and for every $\eta \in \left(0, \frac{1}{\mathrm{card}(G)}\right)$, there exist a simple separable unital AH algebra $A$ with stable rank one and an action $\alpha \colon G \to \mathrm{Aut}(A)$ such that:

1. $\alpha$ is pointwise outer and strictly approximately inner.
2. $\mathrm{rc}(A) = \mathrm{rc}\left(C^*(G, A, \alpha)\right) = \eta$.

## Sketch of the proof (continued).

1. Let $G$ be a (nontrivial) finite group and let $z\colon G \to \mathrm{U}(l^2(G))$ be the left regular representation. Set $\nu = \mathrm{card}(G)$.
2. There is $w \in \mathrm{U}(M_\nu \otimes M_\nu)$ such that, for all $g \in G$,
$$w(z_g \otimes z_g)w^* = z_g \otimes 1_{M_\nu}.$$
3. Let $s(n)$ and $r(n)$ be sequences in $\mathbb{Z}_{>0}$ with $\lim_{n\to\infty} \frac{s(n)}{r(n)} = \eta$.
4. For $n \in \mathbb{Z}_{\geq 0}$, set $X_n = (S^2)^{s(n)}$.
5. For $n \in \mathbb{Z}_{\geq 0}$ and $j = 1, 2, \ldots, d(n+1)$, let $P_j^{(n)}\colon X_{n+1} \to X_n$ be the $j$ coordinate projection.
6. For $n \in \mathbb{Z}_{\geq 0}$, set
$$A_n = C\left(X_n, M_\nu\right) \otimes M_{r(n)}.$$

## Sketch of the proof (continued).

7. For $n \in \mathbb{Z}_{\geq 0}$, define $\alpha^{(n)} \colon G \to \mathrm{Aut}(A_n)$ by

$$\alpha_g^{(n)}(f \otimes c) = \mathrm{Ad}\left(1_{C(X_n)} \otimes z_g\right)(f) \otimes c$$

for $f \in C(X_n, M_\nu)$, $c \in M_{r(n)}$, and $g \in G$.

8. Define $\Gamma_{n+1,\,n} \colon A_n \to A_{n+1}$ by

$$f \otimes c \mapsto \begin{pmatrix} f \circ P_1^{(n)} & & & 0 \\ & \ddots & & \\ & & f \circ P_{d(n+1)}^{(n)} & \\ 0 & & & w(1_{M_\nu} \otimes f(x_n))w^* \end{pmatrix} \otimes c$$

for $f \in C\left(X_n, M_\nu\right)$, $c \in M_{r(n)}$, and $x_n \in X_n$.

9. Set

$$A = \varinjlim (A_n, \Gamma_{n+1,\,n}) \qquad \text{and} \qquad \alpha = \varinjlim \alpha^{(n)}.$$

$\dashrightarrow$

**Sketch of the proof (continued).**

10. $A$ is a simple separable unital AH-algebra with stable rank one.
11. $\alpha$ is strictly approximately inner.
12. $\alpha$ is pointwise outer. Therefore $C^*(G, A, \alpha)$ is simple.
13. We can show that

$$\mathrm{rc}(A) = \mathrm{rc}\big(C^*(G,\, A,\, \alpha)\big) = \eta.$$

■

Thank you for your attention!